MEM6804 Modeling and Simulation for Logistics and Supply Chain: Theory & Analysis

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2021 (full-time)

Assignment 1

Due Date: March 24 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show enough intermediate steps.
(c) Write your answers independently.

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Question 1 ($6 + 4 = 10$ points)

(1) Prove the result in Buffon's Needle: $\mathbb{P}(\text{needle crosses a line}) = \frac{2l}{\pi d}$. (Lec 1 page 24/32)

(2) If the straight needle is bent to V shape, and let X denote the number of intersection points between a needle and the lines, prove that $\mathbb{E}[X] = \frac{2l}{\pi d}$.

Question 2 ($6 \times 5 = 30$ points)

Are the following statements true or false for general case? If true, prove it; otherwise, give a counter-example.

(1) If X is independent of Y, Y is independent of Z, then X is independent of Z.
(2) X and Y are two random variables. If X^2 is independent of Y^2, then X is independent of Y.
(3) X and Y are two independent random variables. Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then, $g(X)$ is independent of $h(Y)$.
(4) If $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$, then X is independent of Y.
(5) $\rho(X, X^2)$ must be nonzero.

Question 3 (5 points)

Prove that $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$.

Question 4 (10 points)

$Y \sim \text{geometric}(p)$, prove that $\mathbb{E}[X] = 1/p$ and $\text{Var}(X) = (1-p)/p^2$. (Do not directly use the property of negative binomial distribution; use the definition.)

Question 5 (10 points)

If $X \sim \text{Poisson}(\lambda)$, prove that $\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$.

Question 6 (10 points)

If $X \sim \text{Exp}(\lambda)$, prove that $\mathbb{E}[X] = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$. (Do not directly use the property of Erlang distribution or Gamma distribution; use the definition.)

Question 7 (10 points)

If $X, Y \sim \text{Exp}(\lambda)$ and they are independent, prove that $X + Y \sim \text{Erlang}(2, \lambda)$. (Do not directly use the property of Erlang distribution or Gamma distribution; use the definition.)

Question 8 (10 points)

If $X \sim \text{Exp}(\lambda)$, prove that $cX \sim \text{Exp}(\lambda/c)$ for $c > 0$. (Do not directly use the property of Erlang distribution or Gamma distribution; use the definition.)

Question 9 (5 points)

If $X_1, X_2, \ldots, X_n$ are n independent random variables, and $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, \ldots, n$, prove that

$$\min\{X_1, \ldots, X_n\} \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n).$$